Table 1 Ignition delay data of RFNA-aniline and ($RFNA + SO_2$)-aniline systems

	Temperature = 20 ± 2°C Ignition delay, sec	
Mixture ratio, O/F	RFNA-aniline	($RFNA + SO_2$)-aniline
0.50	No ignition	1.00
0.84	No ignition	0.60
1.00	No ignition	0.20
1.34	20.9	0.15
1.67	8.0	0.05
2.67	4.8	0.05
3.34	6.6	0.10

It is obvious that ignition delay of this system is considerably reduced with respect to RFNA aniline.

($RFNA + SO_2$) may not be good as an oxidizer since it contains sulphur which is likely to increase the average molecular weight of the combustion products and consequently it may reduce the efficiency of the propellant system, but it would certainly be very useful as oxidizer for auxiliary ignition.[8]

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Nonorthogonal Coordinates

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1.0 Introduction

A VECTOR as defined in classical mechanics is a special quantity which has a magnitude and a direction

$$\bar{x} = x_{1A}\bar{\imath}_A + x_{2A}\bar{\jmath}_A + x_{3A}\bar{k}_A$$
$$= x_{1B}\bar{\imath}_B + x_{2B}\bar{\jmath}_B + x_{3B}\bar{k}_B \qquad (1.1)$$

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where $\bar{\imath}_A, \bar{\jmath}_A, \bar{k}_A$ and $\bar{\imath}_B, \bar{\jmath}_B, \bar{k}_B$ are base vectors associated with coordinate frames A and B respectively. x_{1A}, x_{2A}, x_{3A} and x_{1B}, x_{2B}, x_{3B} are coefficients of the vector x associated with each of the base vectors.

A dyadic is a special quantity having a magnitude and two associated directions defined as follows: $\bar{\bar{I}} = I_{11}\bar{\imath}\bar{\imath} + I_{12}\bar{\imath}\bar{\jmath} + I_{13}\bar{\imath}\bar{k} + I_{21}\bar{\jmath}\bar{\imath} + I_{22}\bar{\jmath}\bar{\jmath} + I_{23}\bar{\jmath}\bar{k} + I_{31}\bar{k}\bar{\imath} + I_{32}\bar{k}\bar{\jmath} + I_{33}\bar{k}\bar{k}$.

A matrix is an array of elements arranged in some systematic manner. The theory of matrices are simply rules established for the mathematical operation of these large arrays. In classical mechanics, one often has to deal with large sets of vector equations. It will be advantageous to apply matrix theory to facilitate rapid and systematic manipulation of these equations. Thus far, most of the application of matrix techniques involved the transformation of all vector and dyadics into one basic coordinate frame so that their numerical values commune. In what follows, slight modifications and additions are made to the basic matrix theory to ease its application in handling vector equations in classical mechanics. Specifically, we shall introduce matrices whose elements are basis vectors. To prevent confusion, a column or row array of elements will be referred to in this paper as column and row matrices. The word vector is reserved exclusively for the notation of a quantity having a magnitude and a direction.

2.0 Basic Definitions and Notation

2.1 Notations

All lower case letters with a bar are used to denote Euclidean vectors, defined on a three dimensional Euclidean Vector Space. Lower case letters with no bar denote scalars. Parentheses are used to denote a 3 × 1 matrix array.

This symbol $(\bar{e}_A)$ is used to denote a set of three base vectors of coordinate system A, namely,

$$(\bar{e}_A)^T = (\bar{e}_{1A},\bar{e}_{2A},\bar{e}_{3A}) \qquad (2.1)$$

Brackets are used to denote a square matrix. The tilde matrix $[\tilde{c}]$ is the 3 × 3 skew-symmetric matrix associated with a particular 3 × 1 row matrix $(c)^T = (c_1,c_2,c_3)$ defined as follows.

$$[\tilde{c}] = \begin{bmatrix} 0 & -c_3 & c_2 \\ c_3 & 0 & -c_1 \\ -c_2 & c_1 & 0 \end{bmatrix} \qquad (2.2)$$

2.2 Special definitions

2.2.1 Matrix representation of a vector. Every vector $\bar{x}$ of a three dimensional vector space can be expressed in terms of a Base Vector Matrix $(\bar{e}_A)$, and a Component Matrix, (x_A), in the form given by Eq. 2.3 as follows:

$$\bar{x} \triangleq (\bar{e}_A)^T(x_A) = (x_A)^T(\bar{e}_A)$$
$$= x_{1A}\bar{e}_{1A} + x_{2A}\bar{e}_{2A} + x_{3A}\bar{e}_{3A} \qquad (2.3)$$

It is seen that Eq. 2.3 is consistent with the Cartesian representation of a vector.

2.2.2 Matrix representation of a dyadic

$$\bar{\bar{I}} = (\bar{e}_A)^T[I_A](\bar{e}_A) \qquad (2.4)$$

Every dyadic can be represented in terms of a Component Matrix, $[I_A]$, and a set of base vectors in a form given by Eq. 2.4. The component matrix $[I_A] = [I_{ijA}]$ is the 3 × 3 matrix formed from the nine components of $\bar{\bar{I}}$ with respect to $\bar{e}_{iA}\bar{e}_{jA}$.

2.3 Fundamental Operations

Two fundamental mathematical operations with the base vector matrices are defined below. These two fundamental operations are the basis for formalizing all algebraic operations with vectors and dyadics.

2.3.1 Dot product matrix

$$[D_{AB}] \triangleq (\bar{e}_A)\cdot(\bar{e}_B)^T$$

where

$$D_{ABij} \triangleq (\bar{e}_{iA})\cdot(\bar{e}_{jB}) \tag{2.5}$$

2.3.2 Cross product matrix

$$[\tilde{C}_{AB}] \triangleq (\bar{e}_A) \times (\bar{e}_B)^T$$
$$C_{ABij} \triangleq \bar{e}_{iA} \times \bar{e}_{jB} \tag{2.6}$$

The cross product matrix, $[\tilde{C}_{AB}]$, is a 3×3 matrix with vector elements.

3.0 Techniques for the Operation of General Nonorthogonal Coordinate Systems

In engineering applications, there are numerous problems which require the use of nonorthogonal systems either because of convenience or hardware constraints. To facilitate the operation in the nonorthogonal coordinate systems, we will first define a reciprocal set of vectors $(\bar{e}^*)$ and which have the properties:

$$\bar{e}_i\cdot\bar{e}_j^* = \delta_{ij} = \begin{cases} = 1 \text{ if } i = j \\ = 0 \text{ if } i \neq j \end{cases} \tag{3.1}$$

where $\bar{e}_i$'s are the basic set of unit vectors in the nonorthogonal coordinate system. Note that the $\bar{e}_j^*$'s are not unit vectors, in fact they obey the following relationships:

$$\bar{e}_j^*\cdot\bar{e}_j \triangleq 1 = |\bar{e}_j^*| \cos\alpha_{jj}$$
$$\therefore |\bar{e}_j^*| = \frac{1}{\cos\alpha_{jj}} \tag{3.2}$$

where α_{jj} = angle between $\bar{e}_j$ and $\bar{e}_j^*$.

3.1 Basic Vector Operations in Nonorthogonal Systems

3.1.1 Coordinate transformation

$$(\bar{e}_A) = [T_{AB}](\bar{e}_B) \tag{3.3}$$

To get $[T_{AB}]$, post dot equation with $(\bar{e}_B^*)^T$. Hence we have

$$[T_{AB}] = (e_A)\cdot(\bar{e}_B^*)^T \text{ Note that } (\bar{e}_B)\cdot(\bar{e}_B^*)^T = U \tag{3.4}$$

Hence the transformation matrix is always obtained by post dotting the set of unit vectors of the 1st frame with the reciprocal set of unit vectors of the 2nd frame.

3.2 Relations Between the Basic Set and the Reciprocal Set of Vectors

To find the reciprocal set, we write

$$(\bar{e}_A^*) = [T_{A^*A}](\bar{e}_A) \tag{3.5}$$

where

$$[T_{A^*A}] \triangleq (\bar{e}_A^*)\cdot(\bar{e}_A^*)^T = [T_{AA^*}]^{-1} = [(\bar{e}_A)\cdot(\bar{e}_A)^T]^{-1}$$

3.2.1 Representation of a vector in a nonorthogonal set

$$\bar{R} \triangleq (\bar{e}_A)^T(R_A^*) \triangleq (\bar{e}_A^*)^T(R_A) \tag{3.6}$$

Where the coefficient vectors are defined as follows:

$$(R_A^*) = \bar{R}\cdot(\bar{e}_A^*),\ (R_A) = \bar{R}\cdot(\bar{e}_A)$$

Note that $(R_A) = [T_{AA^*}](R_A^*)$

3.2.2 Vector Algebra

a) dot product

$$\langle\bar{x},y\rangle = (x_A^*)^T(\bar{e}_A)\cdot(\bar{e}_A)^T(y_A^*) = (x_A^*)^T[T_{AA^*}](y_A^*) = (x_A^*)^T(y_A) \tag{3.7}$$

b) vector cross product

From elementary considerations, we have

$$\bar{e}_1 \times \bar{e}_2 = \bar{n}_3 \sin\theta_{12} \tag{3.8}$$

where n_3 is a unit vector normal to $\bar{e}_1$ and $\bar{e}_2$ and θ_{12} is the angle between $\bar{e}_1$ and $\bar{e}_2$. Using the reciprocal set, we know that $\bar{e}_3^*$ is along the direction of $\bar{n}_3$, however, it is not a unit vector.† Hence we have $\bar{n}_3 = \cos\alpha_{33}\bar{e}_3^*$, $\bar{e}_1 \times \bar{e}_2 = \sin\theta_{12} \cos\alpha_{33}\bar{e}_3^*$. Similarly, $\bar{e}_2 \times \bar{e}_3 = \sin\theta_{23} \cos\alpha_{11}\bar{e}_1^*$, $\bar{e}_3 \times \bar{e}_1 = \sin\theta_{31} \cos\alpha_{22}\bar{e}_2^*$. It can be shown easily that $\sin\theta_{12} \cos\alpha_{33} = \sin\theta_{23} \cos\alpha_{11} = \sin\theta_{31} \cos\alpha_{22} = \bar{e}_1\cdot(\bar{e}_2 \times \bar{e}_3) = c$ = volume of the parallelipiped formed by $\bar{e}_1$, $\bar{e}_2$, $\bar{e}_3$ = constant. Using this relationship, we have

$$(\bar{e}_A) \times (\bar{e}_A) = c[\tilde{e}^*]^T = -c[\tilde{e}^*] \tag{3.9}$$

Using the above relationships, we can now write down the operation of the cross-product in nonorthogonal systems:

$$\bar{x} \times \bar{y} = (x_A^*)^T(\bar{e}_A) \times (\bar{e}_A)^T(y_A^*)$$
$$= c(x_A^*)^T[\tilde{e}_A^*]^T(y_A^*) = c(\bar{e}_A^*)^T[\tilde{x}_A^*](y_A^*) \tag{3.10}$$

3.2.3 Rate of change of a nonorthogonal basis

$$\frac{d\bar{e}_{1A}}{dt} \triangleq \bar{\omega} \times \bar{e}_{1A}$$

$$\bar{\omega} = \omega_1^*\bar{e}_{1A} + \omega_2^*\bar{e}_{2A} + \omega_3^*\bar{e}_{3A} \text{ (angular velocity vector)}$$

where

$$\omega_i^* = \bar{\omega}\cdot\bar{e}_i^*$$

Following our definitions of the cross product

$$d(\bar{e}_A)/dt = c[\tilde{\omega}_A^*]_A{}^T(\bar{e}_A^*) \tag{3.11}$$

Eq. 3.11 is a very important equation in nonorthogonal coordinate systems. In like manner, we can find the rate of change of the reciprocal basis:

$$\frac{d(e_A^*)}{dt} = c^*[\tilde{\omega}_A]_A{}^T(\bar{e}_A) \tag{3.12}$$

where

$$c^* = \bar{e}_3^*\cdot(\bar{e}_1^* \times \bar{e}_2^*)$$

= volume of parallelepiped formed by $\bar{e}_1^*$, $\bar{e}_2^*$ and $\bar{e}_3^*$.

Note also the following relationships: (see Appendix B).

$$cc^* = 1$$
$$[\tilde{\omega}_A^*]_A = (c^*/c)[T_{AA}]^*[\tilde{\omega}_A]_A[T_{AA}^*] \tag{3.13}$$

3.2.4 Rate of change of a transformation matrix:

$$[T_{AB}] \triangleq (\bar{e}_A)\cdot(\bar{e}_B^*)^T$$

$$\therefore \frac{d}{dt}[T_{AB}] = \frac{d}{dt}(\bar{e}_A)\cdot(\bar{e}_B^*)^T + (\bar{e}_A)\cdot\frac{d}{dt}(\bar{e}_B^*)^T$$

$$= c_A[\tilde{\omega}_{AB}]_A[T_{A^*A}][T_{AB}] \tag{3.14}$$

where

$$[\tilde{\omega}_{AB}]_A = [\tilde{\omega}_A^*]_A - [\tilde{\omega}_B^*]_A.$$

= relative angular velocity tilde matrix (see Appendix B).

3.2.5 Rate of Change of a Vector

$$R = (R_A^*)^T(\bar{e}_A)$$
$$d\bar{R}/dt = (d/dt)(R_A^*)^T(\bar{e}_A) + (R_A^*)^T d/dt(\bar{e}_A)$$
$$= (\bar{e}_A)^T[(\dot{R}_A^*) + c[T_{A^*A}][\tilde{\omega}_A^*]_A(R_A^*)] \tag{3.15}$$

† See Eq. (3.2).

Equations 3.1 through 3.15 are all that's necessary to solve any kinemetric and dynamics problem in nonorthogonal coordinates. These equations are also very useful in solving inertial guidance problems using nonorthogonal sensors.

4.0 Specialization to Orthogonal Coordinate Systems

The use of orthonormal coordinate bases has many practical advantages when the problem permits such use. It will be seen that this case permits key simplifications to be made to the two fundamental matrix operations defined in the previous section since the reciprocal basis for a orthonormal set is itself. Hence all the equations derived previously can be used by noting that $c = 1$, $(\bar{e}_A) \equiv (\bar{e}_A{}^*)$.

Appendix A: Rotational Problems in Nonorthogonal Basis

Lets assume that we have a rotating satellite in space, where the rate gyros, and thrusters (moment generators) are located in two different nonorthogonal bases A and B, respectively. Further, lets assume that the inertial tensor $\bar{\bar{I}}$ is given in yet another basis say C. It is desired to find the dynamic equation (differential equation) governing the time history of the gyro outputs.

$$\bar{\bar{I}} = (\bar{e}_C)^T[I_C](\bar{e}_C) \quad \text{(inertial tensor)} \tag{A1}$$

$$\bar{M} = (\bar{e}_B)^T(M_B) \quad \text{(torque equation)} \tag{A2}$$

$$\bar{\omega} = (\omega_A)^T(\bar{e}_A) \quad \text{(angular velocity)} \tag{A3}$$

Now, we can proceed to solve the dynamic problem

$$\begin{aligned} \bar{H} &= \bar{\bar{I}}\cdot\bar{\omega} \\ &= (\bar{e}_C)^T[I_C](\bar{e}_C)\cdot(\bar{e}_A{}^*)^T(\omega_A) \\ &= (\bar{e}_A)^T[T_{CA}]^T[I_C][T_{CA}](\omega_A) \\ &\triangleq (\bar{e}_A)^T(h_A{}^*) \end{aligned} \tag{A4}$$

Applying the 2nd law of Newton, we have

$$\begin{aligned} d\bar{H}/dt &= (\bar{e}_A)^T[(\dot{h}_A{}^*) + c_A[T_{A*A}][\tilde{\omega}_A{}^*]_A(h_A{}^*)] \\ &= (e_A)^T[(\dot{h}_A{}^*) + c_A{}^*[\tilde{\omega}_A]_A[T_{AA}{}^*](h_A{}^*)] \\ &= \bar{M} = (\bar{e}_B)^T(M_B) \end{aligned} \tag{A5}$$

Finally, we have the desired nonlinear differential equation

$$(\dot{h}_A{}^*) + c_A{}^*[\tilde{\omega}_A]_A[T_{AA*}](h_A{}^*) = [T_{BA}]^T(M_B) \tag{A6}$$

where

$$(h_A{}^*) \triangleq [T_{CA}]^T[I_C][T_{CA}](\omega_A)$$

Admittedly, Eq. (A6) is still rather complicated, but it is no more difficult to solve than any other coupled nonlinear 1st order equations. The purpose of this problem is to illustrate how a seemingly very difficult problem can be attacked systematically using the techniques outlined in this paper.

Appendix B: Some Basic Relationships in Nonorthogonal Coordinate Systems

Relationship between c and c*

$c \triangleq (\bar{e}_3)\cdot(\bar{e}_1 \times \bar{e}_2) \triangleq \text{Det}[E]$

= volume of parallelipiped formed by the vectors $\bar{e}_1, \bar{e}_2, \bar{e}_3$, (B1)

Note that the elements E_{11}, E_{12}, E_{13} are the projections of $\bar{e}_1$ onto any arbitrary orthogonal coordinate system.

$$c^* \triangleq (\bar{e}_3{}^*)\cdot(e_1{}^* \times \bar{e}_2{}^*) \triangleq \text{Det}[E^*]$$

Note that

$$[EE^*] = [U]$$

$$\therefore \text{Det}[EE^*] = \text{Det}[E]\text{Det}[E^*] = cc^* = 1$$

Relationship between $[\tilde{w}]$ and $[\tilde{w}^*]$

Since

$$(\bar{e}_A)\cdot(\bar{e}_A{}^*)^T = [U]$$

Differentiating and rearranging, we have

$$[\tilde{\omega}_A{}^*]_A = \frac{c^*}{c} T_{AA*}[\tilde{\omega}_A]_A T_{AA*} \tag{B2}$$

Relative Angular Velocities

The relative angular velocity between frame A and B expressed in frame A is

$$(\bar{\omega}_{AB})_A \triangleq (\bar{\omega}_A)_A - (\bar{\omega}_B)_A$$

$$\therefore [\tilde{\omega}_{AB}]_A = [\tilde{\omega}_A{}^*]_A - [\tilde{\omega}_B{}^*]_A \tag{B3}$$

where

$$\begin{aligned} [\tilde{\omega}_B{}^*]_A{}^T &= (c_B/c_A)[T_{AB}][\tilde{\omega}_B{}^*]_B{}^T[T_{B*B}][T_{BA}][T_{AA*}] \\ &= (c_B{}^*/c_A)[T_{AB}][T_{BB*}][\tilde{\omega}_B]_B{}^T[T_{BA}][T_{AA*}] \end{aligned} \tag{B4}$$

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Reissner's Plate Equations in Polar Coordinates

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In a recent paper by Lehnhoff and Miller[1] the influence of transverse shear on circular plates has been investigated using a method suggested by Goodier. The resulting plate equations in polar coordinates have been solved using Fourier Analysis. In this Note, the effect of shear deformations on circular plates is studied using Reissner's theory. The plate equations thus obtained in polar coordinates have been reduced in general, to the solution of equations governing the lateral deflection and a stress function. Bending moments and shear forces are expressed as functions of lateral deflection and the stress function. As an example, complete solutions are presented for a circular plate subjected to arbitrary lateral load.

Equilibrium equations governing the bending of a plate in polar coordinates are

$$\partial M_r/\partial r + (1/r)\partial M_{r\theta}/\partial\theta + (M_r - M_\theta)/r - Q_r = 0 \tag{1}$$

$$\partial M_{r\theta}/\partial r + (1/r)\partial M_\theta/\partial\theta + 2M_{r\theta}/r - Q_\theta = 0 \tag{2}$$

$$\partial Q_r/\partial r + Q_r/r + (1/r)\partial Q_\theta/\partial\theta + q = 0 \tag{3}$$

where $q(r,\theta)$ is the lateral load acting on the plate. Reissner's

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